Problem 1.43

(a) Prove that the unit vector $\hat{\mathbf{r}}$ of two-dimensional polar coordinates is equal to

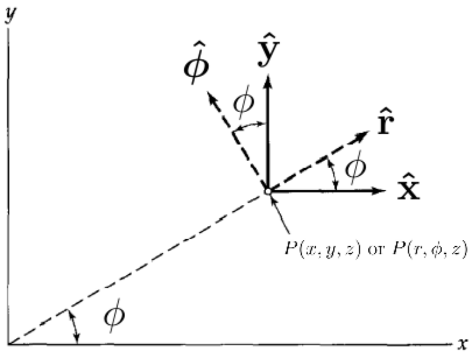
$$\hat{\mathbf{r}} = \hat{\mathbf{x}} \cos\phi + \hat{\mathbf{y}} \sin\phi \tag{1.59}$$

and find a corresponding expression for $\hat{\boldsymbol{\phi}}$. **(b)** Assuming that ϕ depends on the time t, differentiate your answers in part (a) to give an alternative proof of the results (1.42) and (1.46) for the time derivatives $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$.

Solution

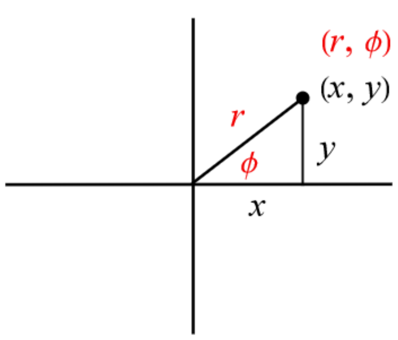
Part (a)

The rectangular (Cartesian) and cylindrical unit vectors in two dimensions are illustrated below.



$\hat{\mathbf{r}}$ is the unit vector of $\mathbf{r}$, the radial position vector. (Note that in spherical coordinates $\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$ because the vector starts from the origin, whereas in cylindrical coordinates $\mathbf{r}$ starts from the z-axis.)

$$\begin{aligned}
\hat{\mathbf{r}} &= \frac{\mathbf{r}}{|\mathbf{r}|} \\
&= \frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + 0^2}} \\
&= \frac{x}{\sqrt{x^2 + y^2}}\,\hat{\mathbf{x}} + \frac{y}{\sqrt{x^2 + y^2}}\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}
\end{aligned}$$



From the Pythagorean theorem, $r = \sqrt{x^2 + y^2}$.

$$\hat{\mathbf{r}} = \frac{x}{r}\,\hat{\mathbf{x}} + \frac{y}{r}\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}$$

Therefore, using the definitions of sine and cosine,

$$\boxed{\hat{\mathbf{r}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}.}$$

$\hat{\boldsymbol{\phi}}$ is perpendicular to both $\hat{\mathbf{z}}$ and $\hat{\mathbf{r}}$. Curling the fingers of the right hand from $\hat{\mathbf{z}}$ to $\hat{\mathbf{r}}$ makes it so the thumb points in the direction of $\hat{\boldsymbol{\phi}}$. By the right-hand corkscrew rule, then,

$$\begin{aligned}
\hat{\boldsymbol{\phi}} &= \hat{\mathbf{z}} \times \hat{\mathbf{r}} \\
&= \hat{\mathbf{z}} \times (\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}) \\
&= \cos\phi\,(\hat{\mathbf{z}} \times \hat{\mathbf{x}}) + \sin\phi\,(\hat{\mathbf{z}} \times \hat{\mathbf{y}}) + 0\,(\hat{\mathbf{z}} \times \hat{\mathbf{z}}) \\
&= \cos\phi\,(\hat{\mathbf{y}}) + \sin\phi\,(-\hat{\mathbf{x}}) + 0\,(\mathbf{0}).
\end{aligned}$$

Therefore,

$$\boxed{\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}.}$$

Part (b)

Differentiate $\hat{\mathbf{r}}$ by using the product rule and the chain rule and noting that the Cartesian unit vectors are independent of time.

$$\begin{aligned}
\frac{d\hat{\mathbf{r}}}{dt} &= \frac{d}{dt}(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}) \\
&= \frac{d}{dt}(\cos\phi\,\hat{\mathbf{x}}) + \frac{d}{dt}(\sin\phi\,\hat{\mathbf{y}}) + \frac{d}{dt}(0\,\hat{\mathbf{z}}) \\
&= \frac{d}{dt}(\cos\phi)\,\hat{\mathbf{x}} + \cos\phi\,\frac{d\hat{\mathbf{x}}}{dt} + \frac{d}{dt}(\sin\phi)\,\hat{\mathbf{y}} + \sin\phi\,\frac{d\hat{\mathbf{y}}}{dt} + 0\,\frac{d\hat{\mathbf{z}}}{dt} \\
&= \frac{d}{dt}(\cos\phi)\,\hat{\mathbf{x}} + \cos\phi\,(\mathbf{0}) + \frac{d}{dt}(\sin\phi)\,\hat{\mathbf{y}} + \sin\phi\,(\mathbf{0}) + 0\,(\mathbf{0}) \\
&= \left[(-\sin\phi)\cdot\frac{d\phi}{dt}\right]\hat{\mathbf{x}} + \left[(\cos\phi)\cdot\frac{d\phi}{dt}\right]\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}} \\
&= \frac{d\phi}{dt}(-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}) \\
&= \frac{d\phi}{dt}\,\hat{\boldsymbol{\phi}}
\end{aligned}$$

This is equation (1.42) on page 27.

Differentiate $\hat{\boldsymbol{\phi}}$ by using the product rule and the chain rule and noting that the Cartesian unit vectors are independent of time.

$$\begin{aligned}
\frac{d\hat{\boldsymbol{\phi}}}{dt} &= \frac{d}{dt}(-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}) \\
&= \frac{d}{dt}(-\sin\phi\,\hat{\mathbf{x}}) + \frac{d}{dt}(\cos\phi\,\hat{\mathbf{y}}) + \frac{d}{dt}(0\,\hat{\mathbf{z}}) \\
&= \frac{d}{dt}(-\sin\phi)\,\hat{\mathbf{x}} - \sin\phi\,\frac{d\hat{\mathbf{x}}}{dt} + \frac{d}{dt}(\cos\phi)\,\hat{\mathbf{y}} + \cos\phi\,\frac{d\hat{\mathbf{y}}}{dt} + 0\,\frac{d\hat{\mathbf{z}}}{dt} \\
&= \frac{d}{dt}(-\sin\phi)\,\hat{\mathbf{x}} - \sin\phi\,(\mathbf{0}) + \frac{d}{dt}(\cos\phi)\,\hat{\mathbf{y}} + \cos\phi\,(\mathbf{0}) + 0\,(\mathbf{0}) \\
&= \left[(-\cos\phi)\cdot\frac{d\phi}{dt}\right]\hat{\mathbf{x}} + \left[(-\sin\phi)\cdot\frac{d\phi}{dt}\right]\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}} \\
&= -\frac{d\phi}{dt}(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}) \\
&= -\frac{d\phi}{dt}\,\hat{\mathbf{r}}
\end{aligned}$$

This is equation (1.46) on page 28.